The Alibaba Global Mathematics Competition (Hangzhou 2018) consists of 3 problems. Each consists of 3 questions: a, b, and c.

This document includes answers for your reference. It is important to note that there are multiple answers to each question. If you submitted different answers, you may still get points. We try to write the answers rather thoroughly. It does not mean your answers need to be as detailed. This document is neither a rubric nor a grading guide. The authors of these answers are not the graders.

(correction 1[1])

Problem 1.

a. During the Alibaba 11.11 Shopping Festival, Store A issues "5 RMB off 60 RMB" stackable coupons. "Stackable" means the multiple coupons can be applied to a single order. For example, an order of 120 RMB at list price, can be reduced to 110 RMB by applying two such coupons.

Store A is part of Tmall.com. Tmall.com issues a "60 RMB off 299 RMB" coupon, limited to one per order. This coupon applies to the list price and is stackable with any individual store coupons. For example, to a product listed at 299 RMB in a Tmall.com store, one pays only 299 RMB - (the store discount based on 299 RMB) - 60 RMB. If the total list price is slightly below 299RMB, customers often adds filler item(s) (such as socks or tissues) from other Tmall.com stores to reach 299RMB and then apply the coupon.

Xiao Ming will buy a 250 RMB pair of headphones and a 600 RMB speaker set from Tmall.com Store A. Xiao Ming has unlimited access to the two types of coupons described. What is the least amount that he must pay?

Answer: 709 RMB. To get this answer, we have used filler items from other store(s). The answer will be reduced to 705 RMB if there are filler items solely from Store A (but this is less likely to hold in practice.) Below, we explain the steps to get 709 RBM.

Below we compare buying both items in one order and buying them in two separate orders. The latter at 709 is cheaper.

Buy the two items in one order: The final cost is

$$\begin{aligned}&250 \text{ (headphones' list price)} + 600 \text{ (speaker set's list price)}\\&- 14 \times 5 \text{ apply } (\lfloor \tfrac{250+600}{60} \rfloor = 14 \text{ "5-off-60" store coupons})\\&- 60 \text{ (shopping cart coupon)}\\&= 720.\end{aligned}$$

Buy the two items separately: The two orders cost 709, which breaks down to the following two orders: The headphone pair costs

$$\begin{aligned}&250 - 4 \times 5 \text{ (apply } \lfloor \tfrac{250}{60} \rfloor = 4 \text{ store coupons)}\\&+ 49 \text{ (filler items to reach 299 total list price)} - 60 \text{ (shopping cart coupon)}\\&= 219.\end{aligned} \tag{1}$$

[1] change log: "50 RMB off 299 RMB" is corrected to "60 RMB off 299 RMB"

If one forgets to use filler items, s/he will pay $250 - 20 = 230$, which is 11 more.

The speaker set costs

$$\begin{aligned}&600 - 10 \times 5 \text{ (apply } \lfloor \tfrac{600}{60} \rfloor = 10 \text{ store coupons)}\\&- 60 \text{ (shopping cart coupon)}\\&= 490.\end{aligned} \tag{2}$$

Hence all together $219 + 490 = 709$ RMB. □

b. You plan to open your own Tmall.com store, called "Store B," selling the same headphones and speaker set at the same list prices as Store A does. Your store sells only these two models.

You plan to issue "x RMB off 99 RMB" coupons, limited to one per order, where x is an integer greater than 0 and smaller than 99. (For example, the discount for an order of 250 RMB is x RMB, not $2x$ RMB). The Tmall.com "60 RMB off 299 RMB" coupon can be applied to purchases at store B and can be stacked with your "x RMB off 99 RMB" coupon.

What is the minimal number x such that Xiao Ming can spend at least 1 RMB less on *either* the 250 RMB pair of the headphones *or* the 600 RMB speakers set in your Store B than in Store A?

What is the minimal number x such that Xiao Ming can spend at least 1 RMB less for buying *both* the 250 RMB pair of the headphones *and* the 600 RMB speakers set in your Store B than in Store A?

To clarify, the comparison is between the costs with the coupons applied optimally.

Answers: 1st question: 21 if using filler items from other stores and 25 if using filler items from Store A; 2nd question: 36 for the 2nd question if using filler items from other stores and 38 if using filler items from Store A. Below, we give the steps assume we use filler items from other stores.

The 1st question. To buy a headphone pair in your store, one pays $250 - x + 49$ (filler) $-$ 60 (shopping cart coupon) $= 239 - x$. Similarly, we get $540 - x$ for the speaker set.

For your store to cost less on the headphone pair, x must satisfy $239 - x \le 219$ (1), or $x \ge 21$.

For your store to cost less on the speaker pair, x must satisfy $540 - x \le 490 - 1$ (2), or $x \ge 51$.

When $x = 21$, we ensure the headphone pair to be cheaper, not the speaker set though.

The 2nd question. To buy both items in your store, it is cheaper to buy them in two separate orders since we can apply the coupon to each order to get a total discount of $2x$.

The part above has the formulas for the two orders: $(239–x)$ and $(540–x)$. Their total must be cheaper than 709, which is the answer in part 2. That is $(239–x) + (540–x) \le 709 - 1$, or $x \ge 35.5$. Since x is an integer, we set $x = 36$ for this question[2]. □

c. *Mathematical modeling of product bundling.* Suppose that the total costs of Item 1 and Item 2 are c_1 and c_2 (including production, storage, transportation, promotion, etc.), respectively. When a customer visits the Tmall.com store, s/he perceives the values of these items at S_1 and S_2, respectively. We suppose that S_1 and S_2 are random variables that are independently and uniformly distributed on the intervals $[0, u_1]$ and $[0, u_2]$, respectively. There are three questions.

[2]Due to different understanding of the Chinese version, both 36 and 51 can be taken as the correct answer, because there, one may understand that one might not have to buy both items in your store or both items in store A.

1. What is the value for p_1, the price for Item 1, that maximizes the expected profit for each visiting customer? Here, assume that a visiting customer will purchase one piece of Item 1 if $S_1 \geq p_1$, and if so, your profit is $(p_1 - c_1)$. Please provide a formula. Similarly, what is the value for p_2 that maximizes the expected profit for each visiting customer?

   Answer: optimal price $p_i^* = \frac{u_i+c_i}{2}$ and expected profit $r_i^* = \frac{(u_i-c_i)^2}{4u_i}$ for $i = 1, 2$. Since the steps are identical for $i = 1, 2$, we drop i for brevity. Let R be the random variable of profit, which depends on S. We calculate its expectation:

$$\begin{aligned} r = \mathbb{E}_S(R) &= \mathbb{E}_S\big((p-c)\delta_{\text{buy}}\big) \\ &= \mathbb{E}_S\big((p-c)\delta_{p\leq S}\big) \\ &= \int_0^u (p-c)\delta_{p\leq s} \cdot \frac{1}{u} ds \\ &= (p-c)\frac{s}{u}\Big|_{s=p}^{s=u} \\ &= \frac{(p-c)(u-p)}{u}. \end{aligned}$$

   Alternatively, we can obtain the same expected profit directly as the product of profit, $(p - c)$, and the probability of buying, $\frac{u-p}{u}$.

   The function $r(p) := \frac{(p-c)(u-p)}{u}$ is a concave quadratic function, so its maximum is attained at the point $p*$ such that $r'(p^*) = 0$ if p^* is on the interval of its allowed values, $[0, u]$. Indeed, $r'(p^*) = 0$ yields $p* = \frac{u+c}{2}$, which is the maximizer if $c \leq u$ (otherwise, $p^* = u$, which is a trivial case).

   With $p^* = \frac{u+c}{2}$, we get $r^* = r(p^*) = \frac{(u-c)^2}{4u}$. □

2. Assume we are going to sell a bundle item including one unit of Item 1 and one unit of Item 2 at price p_{12}. The total cost of this item is $t(c_1 + c_2)$, where $0 < t < 1$. Assume a visiting customer will purchase one piece of this bundle if $(S_1 + S_2) \geq p_{12}$, and if so, your profit is $p_{12} - t(c_1 + c_2)$. Determine the price p_{12} to maximize the expected profit for each visiting customer. Please provide a formula.

   Answer: the price p_{12} that maximizes the expected return is

$$p_{12}^* = \begin{cases} \frac{1}{3}\big(c_{12} + \sqrt{c_{12}^2 + 6u_1u_2}\big), & c_{12} \in [0, \frac{3}{2}u_1 - u_2] \\ \frac{1}{4}\big(u_1 + 2u_2 + 2c_{12}\big), & c_{12} \in [\frac{3}{2}u_1 - u_2, u_2 - \frac{1}{2}u_1] \\ \frac{1}{3}\big(u_1 + u_2 + 2c_{12}\big), & c_{12} \in [u_2 - \frac{1}{2}u_1, u_1 + u_2]. \end{cases}$$

   Note that p_{12}^* is continuous with respect to c_{12}, including one the boundary points of three intervals, so one can include each boundary point in either or both of the neighboring intervals.

   Also note that the calculation is *not* unique. Students can find the right answer by drawing a picture and using geometry.

   No matter which approach is used, it takes the following three steps to compute p_{12}^*.

   Step 1. Define random variable $S_{12} := S_1 + S_2$. Compute the distribution of S_{12}, denoted by p_{12}. This is *not* a uniform distribution.

$$p_{12}(s) := \Pr(S = s) = \int_0^{u_1+u_2} p_1(z-y)p_2(y)dy = \cdots = \begin{cases} \frac{s}{u_1u_2}, & s \in [0, u_1] \\ \frac{1}{u_2}, & s \in [u_1, u_2] \\ \frac{u_1+u_2-s}{u_1u_2}, & s \in [u_2, u_1 + u_2]. \end{cases}$$

Step 2. Compute the expected profit as a function of S_{12}, which is

$$\mathbb{E}_{S_{12}}\big((p_{12}-c_{12})\delta_{\text{buy}}\big) = \left(\int_0^{u_1} \frac{s}{u_1u_2} + \int_{u_1}^{u_2} \frac{1}{u_2} + \int_{u_2}^{u_1+u_2} \frac{u_1+u_2-s}{u_1u_2}\right)(p_{12}-c_{12})\delta_{p_{12}\le s_{12}} ds_{12}$$

$$= \cdots = (p_{12}-c_{12}) \times \begin{cases} 1-\frac{p_{12}^2}{2u_1u_2}, & p_{12}\in[0,u_1] \\ 1-\frac{p_{12}}{u_2}+\frac{u_1}{2u_2}, & p_{12}\in[u_1,u_2] \\ \frac{(u_1+u_2-p_{12})^2}{2u_1u_2}, & p_{12}\in[u_2,u_1+u_2]. \end{cases}$$

For $p_{12}\notin[0,u_1+u_2]$, we have $\mathbb{E}_{S_{12}}\big((p_{12}-c_{12})\delta_{\text{buy}}\big)\le 0$ as long as $c_{12}\ge 0$.

Step 3. Over each of the intervals, maximize the expected profit. That is to find the profit maximizer p_{12}^* within each of the intervals.

For $p_{12}\in[0,u_1]$, setting the derivative of $(p_{12}-c_{12})(1-\frac{p_{12}^2}{2u_1u_2})$ to 0 yields

$$p_{12}^* = \frac{1}{3}\left(c_{12}+\sqrt{c_{12}^2+6u_1u_2}\right).$$

Draw the curve or check the second derivative, and it is easy to see the above p_{12}^* is a maximizer. From $p_{12}^*\le u_1$, we get $c_{12}\le \frac{3}{2}u_1-u_2$, which is the condition under which the above p_{12}^* is the maximizer of the expected profit.

Using similar steps, we obtained p_{12}^* in the other two cases and their corresponding intervals of c_{12}. □

3. If you must choose between selling Items 1 and 2 separately and selling them in a bundle, which one do you choose? Is one strategy always better than the other? Why?

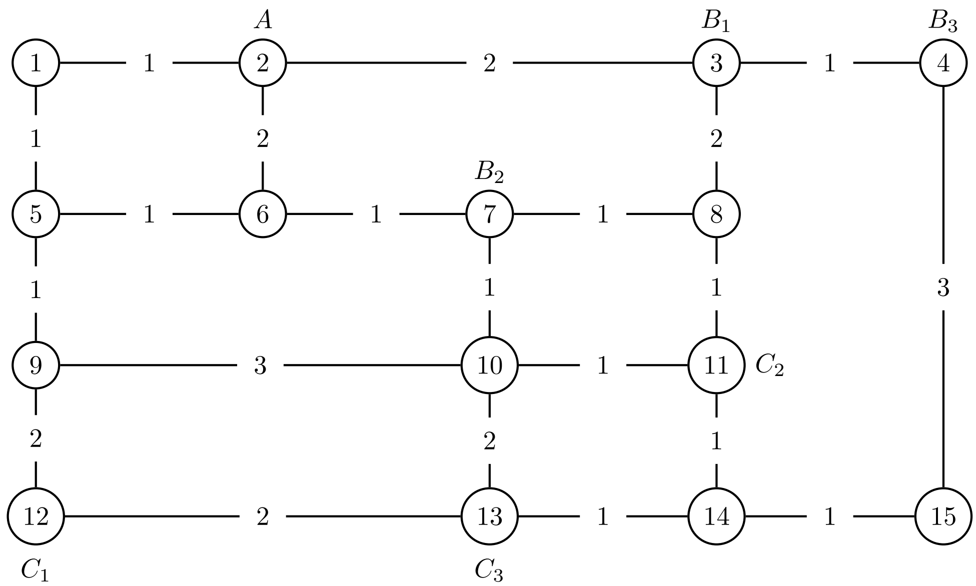
Answer: Neither strategy is always better than the other.

To establish this claim, it is sufficient to use a pair of examples, one showing one strategy better than the other, and the other showing the other way around. There are many such examples, so we do not specify one.

□

Problem 2:

a. The attached figure is an undirected graph. The circled numbers represent the nodes, and the numbers along the edges are their lengths (symmetrical in both directions).



An *Alibaba Hema Xiansheng* carrier starts at point A and will pick up three orders from merchants B_1, B_2, B_3 and deliver them to three customers C_1, C_2, C_3, respectively. The carrier drives a scooter with a trunk that holds at most two orders at any time. All the orders have equal size.

Find the shortest travel route that starts at A and ends at the last delivery. To simplify this question, assume no waiting time during each pickup and delivery.

Answer: The shortest travel distance is 16, attained by the carrier taking the following stops:

$$A \rightsquigarrow B_2 \rightsquigarrow C_2 \rightsquigarrow B_1 \rightsquigarrow B_3 \rightsquigarrow C_3 \rightsquigarrow C_1.$$

There are two slightly different routes with the same length of 16:

Route 1:

$2(A) \to 6 \to 7(B_2) \to 8 \to 11(C_2) \to 8 \to 3(B_1) \to 4(B_3) \to 15 \to 14 \to 13(C_3) \to 12(C_1)$;

Route 2:

$2(A) \to 6 \to 7(B_2) \to 10 \to 11(C_2) \to 8 \to 3(B_1) \to 4(B_3) \to 15 \to 14 \to 13(C_3) \to 12(C_1)$.

Either route will receive the full points.

Enumerating all the full routes and computing their lengths would be exhaustive. However, this problem can be solved without a complete enumeration because the graph is a planar graph and the edge lengths are such that the travel direction is always with 90 degrees of the destination. This means, the shortest path between any two nodes is easy to find.

To solve this problem by hand, one can first guess a good (not necessarily optimal) sequence out of $\{B_1, C_1, B_2, C_2, B_3, C_3\}$ and calculate its travel distance. Indeed, there are several sequences that all lead to a distance of 17. If you get a slightly higher one, it is fine. The distance becomes an upper bound of the shortest distance. Then, start enumerating all the sequences from $\{B_1, C_1, B_2, C_2, B_3, C_3\}$ but eliminate those once a part of their total distance reaches 17. When you find a route with distance 16, it becomes the new upper bound. This procedure is called *branch and bound*. □

b. This question is unrelated to the graph shown in part a; instead, we consider a general graph of many nodes and edges. Suppose that the carrier just picked up an order (we call it the *original order*) and will travel through the edges $e_1, e_2, \ldots, e_m$ in the graph to deliver this original order. When s/he travels through an edge e, s/he may pick up a *new order* for the same destination from a merchant located somewhere on this edge, at probability $P_e \in [0, 1]$. Such probabilities corresponding to the edges $e_1, e_2, \ldots, e_m$ are $P_1, P_2, \ldots, P_m$. We ignore the probability of two or more such new pickups on each edge e as they tend to be very small.

What is the expected number of *new order(s)* for the same destination that this carrier can pick up over the given route (disregarding the trunk capacity)?

What is the probability that s/he picks up at least one *new order* for the same destination over the given route?

Answer: For the 1st question, $P_1 + P_2 + \cdots + P_m$**.** Let $u_i \in \{0, 1\}$ be the number of pickup over edge e_i, for $i = 1, \ldots, m$. We can get our answer from

$$\mathbb{E}\left(u_1 + u_2 + \cdots + u_m\right) = \mathbb{E}(u_1) + \mathbb{E}(u_2) + \cdots + \mathbb{E}(u_m) = P_1 + P_2 + \cdots + P_m.$$

For the 2nd question, $1 - (1 - P_1)(1 - P_2) \ldots (1 - P_m)$**.** Here, $(1 - P_i)$ is the probability of no pickup over e_i and, by statistical independence, $(1 - P_1)(1 - P_2) \ldots (1 - P_m)$ is that of no pickup over the entire route, so 1 minus this product is the probability of picking up at least one order.

Alternatively, one can use conditional probability to obtain the recursion:

$$\begin{aligned}
&P_1 + (1 - P_1)\Pr(\text{at least one new pickup after } e_1) \\
&= P_1 + (1 - P_1)\big(P_2 + (1 - P_2)\Pr(\text{at least one new pickup after } e_2)\big) \\
&= \ldots \\
&= P_1 + (1 - P_1)\Big(P_2 + (1 - P_2)\big(P_3 + \ldots (P_{m-1} + (1 - P_{m-1})P_m))\big)\Big).
\end{aligned}$$

This recursion is also a right answer.

Both of the above answers are also equal to

$$\sum_{k=1}^{m} \left((-1)^{k+1} \sum_{\text{distinct } i_1, \ldots, i_k \in \{1, \ldots, m\}} P_{i_1} P_{i_2} \ldots P_{i_k} \right).$$ □

c. This question is a followup of part b. In this question, we no longer fix the route in part b but find one to maximize the carrier's profit. Suppose that the carrier receives a fixed reward of r for each delivery and spends ℓ, which equals the total lengths of the edges that s/he travels from pickup to delivery. In total, s/he makes a profit of $r - \ell$ on this delivery. (We have set the cost factor of travel distance to 1, for simplicity.)

Suppose that the carrier just picked up the original order and has this order only. What is his/her optimal route assuming the scooter's trunk has a capacity of 2 orders? You shall consider both the travel distance as a cost and the possible additional profit of r for picking up a *new order*. Because any new order has the same destination, its travel cost is ignored. Also, suppose that $0 \leq P_e \leq \min\{\ell_e/r, 1\}$, where ℓ_e is the length of edge e and P_e is defined in part b.

Answer: Assume, without loss of generality, there are T nodes and T is the destination node. First, from every node i, find the shortest path to T and its shortest travel distance c_i (if there is a tie between multiple paths with the same distance, break it arbitrarily.) For $i = T$, we have $c_T = 0$.

Next, using $\{c_1, c_2, \ldots, c_T\}$, compute the optimal expected future reward r_i at every node i, using the maximization formula (3) given below. For $i \neq T$, let j_i be the adjacent node of i that attains the maximum (again, if there is a tie, break it arbitrarily.)

The carrier's best route is decided at each node as follows: at node i, if the carrier has yet to pick up an extra order, travel to j_i; if the carrier has picked up an extra order, then his/her trunk has reached the maximum capacity, so follow the shortest path from i to T.

Note that the above route is not predetermined but decided as the carrier travels. In other word, it is a strategy or policy. It is better than following any predetermined route because the best way depends on whether an extra pickup is made or not.

When the carrier is at a node i and has not made a second pickup, deciding where the carrier should go uses the expectation of the "profit to go", which further depends on both pickup probabilities and travel distance to T.

Define r_i as *the optimal expected future profit at node i* before an extra pickup. For $i = T$, we let $r_T = r$, which is the fixed reward. Suppose we have calculated r_j for the adjacent nodes of i. At i, if we travel to the adjacent node j, then the expected future profit becomes:

- $(2r - c_j) - \ell_{(i,j)}$, if a pickup occurs over (i,j), happening with probability $P_{(i,j)}$;
- $r_j - \ell_{(i,j)}$, if a pickup does not occurs over (i,j), happening with probability $1 - P_{(i,j)}$,

where $\ell_{(i,j)}$ is the length of edge (i,j). The maximum over all the adjacent nodes is

$$\begin{aligned} r_i &= \max_{j \text{ is adjacent to } i} \left\{ P_{(i,j)}((2r - c_j) - \ell_{(i,j)}) + (1 - P_{(i,j)})(r_j - \ell_{(i,j)}) \right\} \\ &= \max_{j \text{ is adjacent to } i} \left\{ (1 - P_{(i,j)})r_j + P_{(i,j)}(2r - c_j) - \ell_{(i,j)} \right\}. \end{aligned} \tag{3}$$

This is known as the Bellman equation.

Given $r_T = r$ and (3), we can compute $\{r_i\}$ using either *dynamic programming*, or more specifically for graphs, Bellman Ford's algorithm or Dijkstra's algorithm (see the justification below). They all start from $r_T = r$ and iteratively determine the elements of $\{r_i\}$.

The condition $P_e \leq \ell_e / r$, or $rP_e \leq \ell_e$, avoids the presence of any "positive reward cycle", which if exists would give the carrier the motivation to cycle around to increase his/her expected reward until an extra order is finally picked up, which is unrealistic.

Note that, Dijkstra's algorithm, which students tend to use over the other choices, requires a "nonnegative edge length" condition. So, if one applies it to compute $\{r_i\}$, they should verify that condition. For our problem, that is to show

$$(1 - P_{(i,j)})r_j + P_{(i,j)}(2r - c_j) - \ell_{(i,j)} \leq r_j. \tag{4}$$

Under the problem assumption $P_e \leq \ell_e / r$, this condition indeed holds. Let us see why. Since the carrier can do no worse than traveling along the shortest path to T (rather than choosing a node that maximizes the expected profit), we have

$$r - c_j \leq r_j,$$

which yields the second inequality in

$$P_{(i,j)}r + (P_{(i,j)}r - \ell_{(i,j)}) \leq P_{(i,j)}r \leq P_{(i,j)}(c_j + r_j),$$

where the first inequality follows from the problem assumption $P_{(i,j)}r \leq \ell_{(i,j)}$. We get (4) by combining the inequalities above. $\square$

Problem 3:

a. Professor Ma has formulated n different but equivalent statements $A_1, A_2, \ldots, A_n$. Every semester, he advises a student to prove an implication $A_i \Rightarrow A_j$, $i \neq j$. This is the dissertation topic of this student. Every semester, he has only one student, and we assume that this student finishes her/his dissertation within the semester. No dissertation should be a direct logical consequence of previously given ones. For example, if $A_i \Rightarrow A_j$ and $A_j \Rightarrow A_k$ have already been used as dissertation topics, Professor Ma cannot use $A_i \Rightarrow A_k$ as a new dissertation topic, as the implication follows from the previous dissertations. What is the *maximal* number of students that Professor Ma can advise?

Answer: We will first construct an answer with $\frac{1}{2}(n+2)(n-1)$ students. Then, we will show this is the best possible answer. We also present an alternative constructive proof that yields $\frac{1}{2}(n+2)(n-1)$.

Construction. First, $(n-1)$ students sequentially prove $A_1 \Rightarrow A_i$ for $i = 2, \ldots, n$. Then, $(n-2)$ students sequentially prove $A_2 \Rightarrow A_i$ for $i = 3, \ldots, n$. Continue this until 1 student proves $A_{n-1} \Rightarrow A_n$. Note that all implications proven so far are valid these and have the form $A_i \Rightarrow A_j$ for $i < j$. Next, $(n-1)$ students sequentially prove $A_n \Rightarrow A_{n-1}, A_{n-1} \Rightarrow A_{n-2}, \cdots, A_2 \Rightarrow A_1$, which are also valid theses. The total number of theses is

$$\big((n-1) + (n-2) + \cdots + 1\big) + (n-1) = \frac{1}{2}n(n-1) + (n-1) = \frac{1}{2}(n+2)(n-1).$$

Let $f(k) := \frac{1}{2}(k+2)(k-1)$.

Proof of optimality. Consider a graph $G = (N, E)$ with nodes $N = \{1, 2, \ldots, n\}$ and directed edges $E = \{(i,j) \mid A_i \Rightarrow A_j \text{ has been shown}\}$. Completing a thesis, i.e., proving an implication, means adding an edge to E.

Define $E' := \{(i,j) \mid A_i \Rightarrow A_j \text{ and } A_j \Rightarrow A_i \text{ have been shown}\} \subseteq E$ be the set of "dual edges." The subgraph $G' = (N, E')$ has at most $2(n-1)$ directed edges; otherwise, there must be a cycle of dual edges, which contains invalid theses.

G has at most $n(n-1)/2$ pairs of nodes. Remove the pairs of nodes with the dual edges, and as we have argued, there are at most $2(n-1)$ directed edges and thus at most $(n-1)$ such pairs, leaving us with $n(n-1)/2 - (n-1) = (n-2)(n-1)/2$ pairs of nodes with either one-way edges or no edge in between. In other words, there are at most $(n-2)(n-1)/2$ one-way edges. Therefore, adding the maximal numbers of one-way and dual edges gives us

$$(n-2)(n-1)/2 + 2(n-1) = \frac{1}{2}(n+2)(n-1) = f(n).$$

Another approach that is a constructive proof. Consider a graph $G = (N, E)$ with nodes $N = \{A_1, A_2, \ldots, A_n\}$ and directed edges $E = \{(A_i, A_j) \mid \text{the statement } A_i \Rightarrow A_j \text{ has been shown}\}$. Completing a thesis $A_i \Rightarrow A_j$ means adding the directed edge (A_i, A_j) to E.

We say A_i implies A_j and write it as $A_i \rightsquigarrow A_j$ if E contains a path (a succession of head-to-tail connected edges) from A_i to A_j.

We say $S \subseteq N$ is a *max equivalent class (MEC)* if $A_i \rightsquigarrow A_j$, for *all* $i, j \in S$, and any larger $S' \supset S$ does have this property. By this definition, if $A_i \rightsquigarrow A_j$ and $j \in S$, then we have $A_i \rightsquigarrow A_k$ for all $k \in S$; hence, we write this as $A_i \rightsquigarrow S$. Similarly, if $A_i \rightsquigarrow A_j$ for some $i \in S$, we write

$S \rightsquigarrow A_j$. Also, if S_1, S_2 are two distinct MECs and $A_i \rightsquigarrow A_j$ for some $i \in S_1,\ j \in S_2$, then we write $S_1 \rightsquigarrow S_2$.

For any MEC S and $i \notin S$, we do *not* have $A_i \rightsquigarrow S$ and $S \rightsquigarrow A_i$ due to the maximality of S.

Depending on E, the set N may be partitioned to the largest MEC, N itself, and at most n individual distinct MECs, $\{1\}, \{2\}, \dots, \{n\}$. Each thesis *may,* but *not necessarily*, join two distinct MECs into their union MEC. Each thesis *cannot* join three or more distinct MECs into the union MEC.

Therefore, our problem reduces to finding the maximal sequence of valid theses that turn the n individual MECs $\{1\}, \{2\}, \dots, \{n\}$ into an MEC of size n.

For integer $x > 0$, let $f(x)$ be the maximal number of sequentially valid theses that generates an MEC of size x, starting from x individual distinct MECs. Clearly, $f(1) = 0$.

For any $n \geq 2$, one must form an MEC of size n by joining two MECs of sizes x and $n - x$, for some $x \in \{1, \dots, n-1\}$. Before the two MECs are joined, there are at most $x(n-x)$ same-way edges between them, and an opposite-way edge completes their joining. Therefore,

$$f(n) = \max_{x \in \{1,\dots,n-1\}} \{f(x) + f(n-x) + x(n-x) + 1\}.$$

Starting from $f(1) = 0$, we can use this formula to compute $f(2), f(3), \dots$. Calculation yields

$$f(n) = \frac{1}{2}(n+2)(n-1).$$

For each n, the term that attains the maximum (say at $x = x_n$) implies an optimal construction: given two subsets of nodes of sizes x_n and $y - x_n$, first add $x_n(y - x_n)$ edges from one subset (either one) to the other, then add edges within each of the two subsets to turn them into MECs, and finally add one opposite-way edge. Those $x_n(y - x_n)$ edges need to be added first since, recall, given two MECs S_1, S_2, adding an edge from one to the other, say S_1 to S_2, establishes $A_i \rightsquigarrow A_j$ for all $i \in S_1, j \in S_2$ and, therefore, prevents the adding of all other edges like (A_i, A_j).

Interestingly, every $x = 1, 2, \dots, n-1$ attains the maximum, implying largely many ways to construct an MEC of size n by $f(n)$ theses. $\square$

b. Let H be an $n \times n$ matrix whose entries are 1 or -1 and whose rows are mutually orthogonal (that is, the standard inner product of every pair of different rows of H is 0). Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Answer: This question is taken from Putnam 2005, question A4.

It is a direct consequence of the following basic result of matrix: For any matrix A of a rows and b columns, we have

$$\text{rank}(A) \cdot \|A\|^2 \geq \|A\|_F^2, \tag{5}$$

where $\|A\|$ is the spectral norm of A and $\|A\|_F$ is the Frobenius norm of A.

Since H in our question is orthogonal and every row of it has norm $\sqrt{n}$, we have $\|H\| = \sqrt{n}$. For any submatrix A of H, $\sqrt{n} = \|H\| \geq \|A\|$. When the entries of A are all 1 and A has a rows and b rows, we have $\|A\|_F = ab$ and $\text{rank}(A) = 1$. Therefore,

$$n \geq \text{rank}(A) \cdot \|A\|^2 \geq \|A\|_F^2 = ab.$$

The question also appears as, for example, Corollary 2.2 in Lokam's "Spectral methos for matrix rigidity ..." J. Computer and System Sciences 64, 449–473, 2001. □

c. Let G be a group with unit element denoted by e. Define the following subset of G:

$$F := \{h \in G \mid h^m = e \text{ for some integer } m \geq 1\}.$$

Show that if F is finite, then there exists an integer $n \geq 1$ such that

$$g^n h = hg^n \quad \text{for all } g \in G,\ h \in F.$$

Proof: Take $g \in G$. Let m_h be such that $h^{m_h} = e$. Let $F_g := \{ghg^{-1} : h \in F\}$. Since $(ghg^{-1})^{m_h} = gh^{m_h}g^{-1} = gg^{-1} = e$, we have $g^{-1}hg \in F$ by the definition of F. Therefore, $F_g \subseteq F$ and $|F_g| \leq |F|$. The same holds for $F_{g^2}, F_{g^3}, \ldots$. This and the finiteness of F imply that, for each h, there exists $\ell_h \leq |F|$ such that $g^{\ell_h}hg^{-\ell_h} = h$. Now take $n = |F|!$ (the factorial of $|F|$). For any $h \in F$, n is a multiple of ℓ_h, that is, $n = l_h \cdot k_h$, so from $g^{\ell_h}hg^{-\ell_h} = h$, we apply cancellations as

$$g^n hg^{-n} = (g^{\ell_h})^{k_h} h (g^{-\ell_h})^{k_h} = (g^{\ell_h})^{k_h - 1} h (g^{-\ell_h})^{k_h - 1} = \cdots = g^{\ell_h} hg^{-\ell_h} = h,$$

from which we get $g^n h = hg^n$. □